Application of Singularity Expansion Method (SEM) to non-uniform transmission lines

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Outline

Introduction: Singularity Expansion Method (SEM)

Theory:

SEM and Transmission Line Approximation

• SEM and asymptotic approach: short and open-circuited lines

• SEM and method of modal parameters

• Measurements and numerical simulation

Conclusion
Introduction

• Different numerical methods (MoM, FDTD, etc.) can be used to calculate induced currents and voltages in wiring systems, but they are not very helpful to gain insight into the physics of coupling phenomena, especially in time domain.

• In contrast, the analytical Singularity Expansion Method (SEM) represents the scattering objects as a set of oscillators, thus giving a physically transparent tool for the coupling phenomena, both in frequency and time domain.

• The SEM was introduced by Carl E. Baum and further developed by Tesche, Giri, Marin, Liu and many other brilliant physicists and engineers.

• Recently, this method has attracted an increasing interest (Meyer, Sandler and Wu) in connection with the definition of the complex eigen-frequencies of a finite straight wire for target identification.
Introduction (cont.)

• The set of SEM poles yields the main contribution for the response function (functional) of the transmission line to an excitation. It also defines the scattering amplitude, response in the time domain, etc.

• So far, all analytical investigations of thin wires were restricted by objects of simple geometric forms or by assuming TL theory.

• In this work, we consider a long non-uniform loaded transmission line above conducting ground illuminated by an incident high-frequency plane-wave.

• To obtain the SEM poles for long finite straight wire, the earlier developed asymptotic approach is used. The zeros of the denominator of this expression yield the SEM poles of the first layer.

• To obtain the SEM poles for a non-uniform wire of arbitrary form, the earlier developed method of modal parameters is used. The determinant of the modal impedance matrix yields the SEM poles in an arbitrary layer.

• The theoretical results have a good agreement with numerical simulation and measurements.
Introduction: SEM

Example: RLC oscillating circuit

- Equation of the circuit:
  \[ I(j\omega) = \frac{V(j\omega)}{j\omega L + 1/j\omega C + R} = V(j\omega)K(j\omega) \]

- Factorization of the transfer function:
  \[ K(j\omega) = \frac{j\omega C}{1-\omega^2 LC + j\omega CR} = -\frac{j\omega C}{(\omega - \omega_1)(\omega - \omega_2)} \]
  \[ \omega_{1,2} = j\frac{R}{2L} \pm \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = j\gamma \pm \tilde{\omega}_0 \]

- Time domain:
  \[ I(t) = \int_{-\infty}^{t} V(t)K(t-t')dt' \]
  \[ K(t) = (\hat{F})^{-1} K(j\omega) = \frac{C}{\omega_2 - \omega_1} \left[ \omega_1 e^{j\omega_1 t} + \omega_2 e^{j\omega_2 t} \right] h(t) = Ce^{-\gamma t} \left( \cos \tilde{\omega}_0 t - \frac{\gamma}{\tilde{\omega}_0} \sin \tilde{\omega}_0 t \right) h(t) \]

This approach was generalized for an arbitrary system by Baum and Tesche, Giri, Marin, Liu. SEM (Singularity Expansion Method) represents the scattering objects as a set of oscillators with complex frequencies \( \omega_n \).
Classical TL: Pulsed excitation of a transmission line

Plane-wave excitation of the symmetrical line, normal incidence

\[ E_z(z, x) = E_0(j \omega) \cdot 2 \sin(kx) \]

\[ \frac{dU}{dz} + j \omega L'I = E_z(x, z) \]

\[ \frac{dI}{dz} + j \omega C'U = 0 \]

\[ U(-L/2) = -Z \cdot I(-L/2) \]

\[ U(L/2) = Z \cdot I(L/2) \]

\[ I(z, j \omega) = \frac{E_z}{j \omega L'} \left[ 1 - \frac{Z \cos(kz)}{jZ_C \sin(kL/2) + Z \cos(kL/2)} \right] \]

\[ I(z, j \omega) = \frac{E_z(j \omega)}{j \omega L'} \left[ 1 - \frac{Z}{Z + Z_C} \frac{e^{-jk(L/2-z)} + e^{-jk(L/2+z)}}{1 - \rho e^{-jkL}} \right] \]

Solution in frequency domain

\[ Z_C = \sqrt{L'/C'} \]

\[ \rho = \frac{Z_C - Z}{Z_C + Z} \]

\[ L' = \frac{\mu_0}{2\pi} \ln(2h/r_0) \]

\[ C' = \frac{2\pi \varepsilon_0}{\ln(2h/r_0)} \]
Solution in time domain

\[ I(z, j\omega) = E_z(j\omega) \cdot K^{TL}(z, j\omega) \]

\[ \Rightarrow I(z,t) = \int_{-\infty}^{t} E_z(t',h) \cdot K^{TL}(t-t',z)dt' \]

\[ K^{TL}(t,z) = 0 \quad \text{for} \quad t \leq 0 \quad \text{- Green's function in time domain} \]

\[ K^{TL}(j \omega, z) = \frac{1}{j \omega L'} \left[ 1 - \frac{Z}{Z + Z_C} \frac{e^{-jk(L/2-z)} + e^{-jk(L/2+z)}}{1 - \rho e^{-jkL}} \right] \quad \rho = \frac{Z_C - Z}{Z_C + Z} \]

To derive \( K(t,z) \) different methods can be used: 1. Numerical inverse Fourier transform; 2. Reflection method; 3. Residue theorem.

Reflection method

\[ (1 - \rho \exp(-jkL))^{-1} = \sum_{n=0}^{\infty} \rho^n \exp(-jknL) \quad \Rightarrow \quad h(t) - \text{Heaviside function} \]

\[ K^{TL}(t,z) = \frac{1}{L'} \left\{ h(t) - \frac{Z}{Z + Z_C} \sum_{n=0}^{\infty} \rho^n \left[ h(t - ((n+1/2) L - z)/c) + h(t - ((n+1/2) L + z)/c) \right] \right\} \]

For short pulses and early times the reflection method converges quite quickly. However, for periodical or quasi-periodical excitations the use of the residue theorem is more convenient.
Solution in time domain. Residue theorem.

Pole for the infinitely long line: \( \omega_\infty = j \frac{R'}{L'} \), \( R' \to 0 \)

Open-circuit – like case \( Z = R > Z_C \) or \(-1 < \rho < 0 \) \( \Rightarrow \)

\( \omega_n = \frac{c}{L} (2\pi(n+1/2) + j \ln(-1/ \rho)) = \omega_n' + j \gamma_n \)

- eigenmodes of finite conductor in TL approximation

\[
K^{TL}(t,z) = \frac{1}{L'} \left[ h(t) - \frac{1}{2} \left[ h(t - L/2c + z/c) + h(t - L/2c - z/c) \right] \right] - \\
- \frac{2Z}{Z + Z_C} \frac{c}{L} \sum_{n=0}^\infty \text{Im} \left\{ \frac{1}{\omega_n L'} \left[ e^{j\omega_n (t-L/2c+z/c)} h(t - L/2c + z/c) + e^{j\omega_n (t-L/2c-z/c)} h(t - L/2c - z/c) \right] \right\}
\]

For times \( t > \max[(L/2+z)/c,(L/2-z)/c] \)

\[
K^{TL}(t,z) = -\frac{2Z}{Z + Z_C} \frac{c}{L} \sum_{n=0}^\infty \text{Im} \left\{ \frac{1}{\omega_n L'} \left[ e^{j\omega_n (t-L/2c+z/c)} + e^{j\omega_n (t-L/2c-z/c)} \right] \right\} = \sum_{n=0}^\infty K_n^{TL}(t,z)
\]
Solution in time domain. TL as a set of oscillators.

For small damping parameters: \( \gamma_n << \omega_n', \quad Z >> Z_C, \quad \rho \approx -1 \)

\[
K_n^{TL}(t,z) = -\frac{4Z}{Z + Z_C} \frac{c e^{-\gamma_n t}}{L} \cdot \sin(\omega_n'(t - L/c)) \cdot \cos(\omega_n'z/c) \]

Current in the TL can be represented as a sum of an infinite number of oscillators. This common property is inherent in all resonating systems.

In time domain each component of the current \( I_n \) satisfy a second order ordinary differential equation:

\[
\ddot{I}_n(z,t) + 2\gamma_n \dot{I}_n(z,t) + \omega_n^2 I_n(z,t) = 0
\]

For the considered simple case these equations are linear and decoupled. But it is not valid in general case, for example for a non-linearly loaded line.
Numerical examples

$L=10 \text{ m}, \ r_0=1\text{ cm}, \ h=0.5\text{ m}$

$Z_c=273 \ \Omega, \ Z_1=Z_2=Z=2000 \ \Omega, \ Z_c << Z$

Eigen frequencies: $f_0=15 \text{ MHz}, \ f_1=45 \text{ MHz}, \ f_2=75 \text{ MHz}, \ f_3=105 \text{ MHz}$

$\gamma_n/2\pi=1.3 \text{ MHz} << f_n$

**Double-exponential pulse:**

$$E_0(t) = E_0 Q(e^{-\alpha t} - e^{-\beta t}) h(t)$$

$E_0=1\text{ V/m}, \ Q=1.05, \ \alpha=10^7 \text{ s}^{-1}, \ \beta=10^8 \text{ s}^{-1}$

**Gaussian pulse:**

$$E_0(t) = E_0 \exp(-t^2/\tau^2)$$

$E_0=1\text{ V/m}, \ \tau=10 \text{ ns}$
Theory I: SEM and asymptotic approach for long lines

• Assumptions:
  - thin wire
  - long line
  - PEC ground plane and wire

• Decomposition of the line into three zones:
  - I and III – near terminal regions with leaky, radiation and TEM modes
  - II – central, asymptotic region:

\[ I(z) = I_0 \exp(-jk_1z) + I_1 \exp(jkz) + I_2 \exp(-jkz) \]

- \( I_0 \) nonhomogeneous–solution for infinite wire
- \( I_1 \) forward and backward running TEM waves
Asymptotic approach for long lines (cont).

\[ I(z) = I_0 \exp(-jk_1z) + I_1 \exp(jkz) + I_2 \exp(-jkz) \]

\[ I_0(j\omega) = \frac{1}{j\omega \sin^2 \theta} \cdot \frac{E_{z}^{\text{ex}}(j\omega,h)}{\mu_0/4\pi \cdot G(j\omega,\theta)} \]

- current amplitude induced by a plane wave in the infinite line

\[ E_{z}^{\text{ex}}(j\omega,h) = E_{z}^{\text{in}} + E_{z}^{\text{ref}} = E^i \sin \theta \cdot (1 - e^{-2jkh\sin \theta}) \]

- amplitude of exciting field

\[ G(j\omega,\theta) = -j\pi[H_0^{(2)}(ka \sin \theta) - H_0^{(2)}(2kh \sin \theta)] \]

\( H \) - Hankel function of zero order and second kind

\[ I_1(j\omega) = I_0 \cdot \frac{C_- \exp(-jkL - jk_1L) + C_+ \rho_- \exp(-2jkL)}{1 - \rho_+ \rho_- \exp(-2jkL)} \]

\[ I_2(j\omega) = I_0 \cdot \frac{C_+ + C_- \rho_+ \exp(-jkL - jk_1L)}{1 - \rho_+ \rho_- \exp(-2jkL)} \]

\( \rho_+, \rho_- \) and \( C_+, C_- \) are reflection and scattering coefficients of the current waves for a semi-infinite line.
Asymptotic approach. Reflection and scattering coefficients

Right semi-infinite line: $0 \leq z$

Non-homogeneous problem (excitation by plane wave):

$I^e_+(z) = I_0 \cdot \Psi^e_+(z)$

$$
\Psi^e_+(z) = \begin{cases} 
\text{exact solution} & 0 \leq z \leq l_{\text{bound}} \\
 e^{-jkz} + C_+(j\omega, \theta)e^{-jkz} & z >> l_{\text{bound}} 
\end{cases}
$$

Homogeneous problem (lumped excitation on the $+\infty$):

$I^0_+(z) = \Psi^0_+(z)$

$$
\Psi^0_+(z) = \begin{cases} 
\text{exact solution} & 0 \leq z \leq l_{\text{bound}} \\
 e^{jkz} + \rho_+(j\omega)e^{-jkz} & z >> l_{\text{bound}} 
\end{cases}
$$

$C$- and $\rho$- are defined in the same way for the left semi-infinite line $-\infty < z \leq 0$.

For low frequencies $kh << 1$ the coefficients coincide with TL solutions. For high frequencies $kh \sim 1$ the coefficients can be found analytically for simple configurations or numerically for more complex configurations.
Low-frequency approximation $kh \ll 1$ - classical TL

$$I_0 = \frac{E_z^e(h)}{j\omega L'_0} \cdot \frac{k^2}{k^2 - k_1^2} = \frac{1}{\sin^2(\theta)} \frac{E_z^e(h)}{j\omega L'_0}$$

$$L'_0 = \mu_0 / 2\pi \cdot \ln(2h/r_0)$$

$$Z_C = \eta_0 / 2\pi \cdot \ln(2h/r_0)$$

$$R^TL_+ = \frac{Z_C - Z_1}{Z_C + Z_1}; \quad R^TL_- = \frac{Z_C - Z_2}{Z_C + Z_2};$$

- reflection coefficients of the “eigen” (TEM) current waves $e^{\pm jkz}$ from the terminals

$$U_1 := \int_0^h E_x^e(0,x)dx; \quad U_2 := \int_0^h E_x^e(0,x)dx$$

$$C^TL_+ = -\frac{Z_1 + Z_C \cos(\theta)}{Z_C + Z_1} + \frac{U_1}{I_0} \frac{1}{Z_C + Z_1}$$

$$C^TL_- = -\frac{Z_2 - Z_C \cos(\theta)}{Z_C + Z_2} + \frac{U_2}{I_0} \frac{1}{Z_C + Z_2}$$

- “scattering coefficients” of the initial current wave $e^{-jk_1z}$ from the terminals

For low frequencies $kh \ll 1$ the three terms-approximation yield classical TL solution
Definition of Reflection Coefficients for Current Waves.

Perturbation theory. (Tkachenko, Rachidi, Nitsch, Ianoz)

The curve of the wire: \( \vec{r}(l) = (x(l), y(l), z(l)) \in \{l\} \)

The unit tangential vector: \( \vec{e}_t(l) = \partial \vec{r}(l) / \partial l \)

\( l \) – is a natural length parameter along the wire

\( E^e_l \) - is the tangential component of the exciting electric field;

\( \varphi(l) \) - is the scalar potential along the wire (in the Lorenz gauge);

Boundary conditions:

\[ I(0) = 0 \quad - \text{for the open-circuited wire}; \]
\[ dI(l)/dl \bigg|_{l=0} = 0 \quad - \text{for the short-circuited wire}; \]

- scalar Greens functions for vector and scalar potentials

\[ g^C(l, l') = \frac{e^{-jk\sqrt{(\vec{r}(l)-\vec{r}(l'))^2 + r_0^2}}}{\sqrt{(\vec{r}(l)-\vec{r}(l'))^2 + r_0^2}} - \frac{e^{-jk\sqrt{(\vec{r}(l)-\vec{r}(l'))^2 + r_0^2}}}{\sqrt{(\vec{r}(l)-\vec{r}(l'))^2 + r_0^2}} \]

\[ g^L(l, l') = \vec{e}_t(l) \cdot \vec{e}_t(l') \frac{e^{-jk\sqrt{(\vec{r}(l)-\vec{r}(l'))^2 + r_0^2}}}{\sqrt{(\vec{r}(l)-\vec{r}(l'))^2 + r_0^2}} - \vec{e}_t(l) \cdot \vec{e}_t(l') \frac{e^{-jk\sqrt{(\vec{r}(l)-\vec{r}(l'))^2 + r_0^2}}}{\sqrt{(\vec{r}(l)-\vec{r}(l'))^2 + r_0^2}} \]
Perturbation theory (cont). MPIE and iterative equations.

Re-write the MPIE equations in a form that is more convenient for the iteration procedure:

\[
\begin{align*}
\frac{\partial \phi(l)}{\partial l} + j\omega L' I(l) &= E_l^0(l) - j\omega \frac{\mu_0}{4\pi} \left[ \int_0^\infty g^{l'}(l,l') I(l') dl' - 2 \ln \left( \frac{2h}{r_0} \right) I(l) \right] \\
\frac{\partial I(l)}{\partial l} + j\omega C' \phi(l) &= - \frac{1}{2 \ln(2h/a)} \left[ \int_0^\infty g^c(l,l') \frac{\partial I(l')}{\partial l'} dl' - 2 \ln \left( \frac{2h}{r_0} \right) \frac{\partial I(l)}{\partial l} \right]
\end{align*}
\]

Write the solution as a sum:

\[
\begin{align*}
I(l) &= I^{(0)}(l) + I^{(1)}(l) + \ldots + I^{(n)}(l) + \ldots \\
\phi(l) &= \phi^{(0)}(l) + \phi^{(1)}(l) + \ldots + \phi^{(n)}(l) + \ldots
\end{align*}
\]

The following elements are small:

\[
I^{(n)}(l), \phi^{(n)} \sim \frac{1}{(2 \ln(2h/a))^n} \quad n \geq 1
\]
Asymptotic approach and SEM poles

For simplicity we consider a case of normal incidence of the plane wave on the horizontal long transmission line with equal loads.

The system of poles of response function $K(z,j\omega)$ includes transversal poles – roots of $G(j\omega)$.

If the frequencies are relatively small $kh<\sim1$ and times are relatively long, one can neglect the input of transversal poles and for the integration along the branch cut one can assume $G(j\omega)\approx 2\ln(2h/r_0)$.

SEM poles of the first layer is given by zeros of the denominator:

This equation can be solved by approximate analytical method or numerically.
Reflection and scattering coefficients

\[ R_+ = R_- = \tilde{R}_+ e^{-2 j k h} \quad \tilde{R}_+^{(0)} = 1 \]
\[ \tilde{R}_+ = \tilde{R}_+^{(0)} + \tilde{R}_+^{(1)} + \tilde{R}_+^{(2)} + \ldots \quad R_+ \approx (1 + \tilde{R}_+^{(1)}) e^{-2 j k h} \]

\[ \tilde{R}_+^{(1)} = \frac{j k}{\ln(2 h / a)} \int_0^\infty \int_0^\infty d l d l' \left\{ - \left[ g^L (l, l') - g^L_{\text{Reg}} (l, l') \right] \cos k l \cos k l' + \left[ g^C (l, l') - g^C_{\text{Reg}} (l, l') \right] \sin k l \sin k l' \right\} \]

\[ g^L_{\text{Reg}} (l, l') = g_0 (l - l', k) + g_0 (l + l', k) \]
\[ g^C_{\text{Reg}} (l, l') = g_0 (l - l', k) - g_0 (l + l', k) \]

\[ C_+ (k, \pi / 2) = C_- (k, \pi / 2) \approx C_+^{(0)} (k, \pi / 2) = - j e^{-j k h} \sin k h \]

Reflection coefficient for the short-circuit vertical riser: \( r_0 = 1 \) cm, \( h = 0.5 \) m

\[ 1 / 2 \ln(2 h / r_0) \] - small parameters of the iterations

Reflection and scattering coefficients change at the characteristic frequency \( \sim \pi c / 2 h \)
Short-circuit line.

Response in frequency domain and SEM poles.

Calculation of SEM poles

**Analytical:**

\[
1 - (1 + \tilde{R}_+^{(1)}(k_n)) \exp(-jk_n(L + 2h)) = 0
\]

\[
k_n = \frac{2\pi n}{L + 2h} - \frac{j}{L + 2h} \ln\left(1 + \tilde{R}_+^{(1)}(k_n)\right)
\]

**Numerical** (Miller 1998):

Pade approximation of the NEC response function in the frequency interval \(0 < \omega < \omega_{\text{max}}\) and calculation the poles by numerical methods.
Short-circuit line. Response in time domain

\[
I(z,t) = \int_{-\infty}^{t} E^\text{in}_{z}(t',h) \cdot K(t-t',z)dt'
\]

\[
K(z,t) = \frac{c}{Z_c} \left[ h(t) - \frac{h}{L+2h} \left[ h(t-(z+h)/c) + h(t-(L+h-z)/c) \right] \right] + \\
+ 2 \text{Re} \left\{ \sum_{n=1}^{\infty} \frac{1}{jk'_n - k''_n} \cdot \frac{1}{\mu_0 G(k'_n)} \cdot \frac{1}{L+2h} \cdot \frac{1}{d\tilde{R}_+(k'_n)} \cdot \frac{d\tilde{R}_+(k'_n)}{dk} \right\}_{k=k'_n} \\
\left[ \exp\left( (jk'_n - k''_n) \cdot c \cdot (t-(L-z)/c) \cdot h(t-(L-z)/c) \right) + \exp\left( (jk'_n - k''_n) \cdot c \cdot (t-z/c) \cdot h(t-z/c) \right) \right]
\]

Response of the short circuit line with two risers:
L=20 m, h=0.5 m, \( r_0=1 \text{mm} \), z=10 m.
Excitation: Gaussian pulse with \( \tau=3 \text{ ns} \)
Open-circuit line. Reflection and scattering coefficients

Reflection and scattering coefficients change at the characteristic frequency \( \sim \pi c/2h \)

\[
\gamma = 0.577
\]

\[
R_+ (j\omega) \approx -1 + j \frac{Si(2kh)}{\ln(2h/a)} + \gamma + \ln(2kh) - Ci(2kh)
\]

\[
\gamma = 0.577
\]

Reflection and scattering coefficients change at the characteristic frequency \( \sim \pi c/2h \)

\[
C_+ (j\omega, \theta) \approx -1 + D_1 + D_2
\]

\[
D_1 \approx \frac{1}{2\ln(2h/a)} \int_0^\infty \exp(-j k \xi) \left[ \left\{ \frac{\exp(j k \xi)}{\xi} - \frac{\exp(-j k \xi \sqrt{\xi^2 + 4h^2})}{\sqrt{\xi^2 + 4h^2}} \right\} \right] d\xi
\]

\[
D_2 = \frac{\pi j}{2\ln(2h/a)} \left[ 1 - \frac{2j}{\pi} \ln(\gamma kh \sin \theta) - H_0^{(2)}(2kh \sin \theta) \right]
\]

\[
C_-(j\omega, \pi - \theta) \approx -1 + D_1 + D_2
\]
Open circuit line. Response in frequency domain.

Parameters:
L=10 m
a=1 cm
h=0.5 m
z=5 m

For simplicity we consider a case of normal incidence of the plane wave on the horizontal long transmission line with equal loads.

\[
I(z, j\omega) = \frac{E_z(j\omega)}{j\omega} \cdot \frac{4\pi}{\mu_0 G(k)} \left[ 1 + \frac{C_+(k, \pi/2) e^{-j\omega L/2} \cos(k(z-L/2))}{1 - \rho(k) e^{-j\omega L}} \right] = E_z(j\omega) \cdot K(z, j\omega)
\]

Response function \(K(j\omega)\) in frequency domain calculated by NEC and the asymptotic approach.

Definition of the SEM poles “by hand”.
Open circuit line, system of SEM poles of function $K(z, j\omega)$.

The system of poles of response function $K(z, j\omega)$ includes transversal poles – roots of

$$G(j\omega_n) = -j\pi \left[ H_0^{(2)}(k_n a) - H_0^{(2)}(2k_n h) \right] = 0$$

and longitudinal poles – roots of the denominator $1 - \rho(k_n)e^{-jk_nL} = 0$

or:

$$k_n = \frac{2\pi}{L} (n + 1/2) - \frac{j}{L} \ln(-\rho(k_n))$$

If the frequencies are relative small $kh < \sim 1$ and times are relative long, one can neglect the input of transversal poles and the integration along branch cut and can assume $G(j\omega) \approx 2\ln(2h/r_0)$

To obtain the “longitudinal” poles one can use an iteration method, using the smallness of the parameter $1/2\ln(2h/r_0)$

$$\rho \approx -1 \quad \Rightarrow \quad k^{(0)} = \frac{2\pi}{L} (n + 1/2) \quad \Rightarrow \quad k^{(1)} = \frac{2\pi}{L} (n + 1/2) - \frac{j}{L} \ln(-\rho(k^{(0)})) \Rightarrow$$

$$k^{(1)}_n \approx \frac{2\pi}{L} (n+1/2) - \frac{Si(2k^{(0)}_n h)}{L \ln(2h/r_0)} + j\gamma + \ln(2k^{(0)}_n h) - Ci(2k^{(0)}_n h)$$

- SEM poles
Open circuit line. Response in time domain.

\[ I(z, j\omega) = E_z^{in}(j\omega) \cdot K(z, j\omega) \xrightarrow{\text{convolution theorem}} I(z, t) = \int_{-\infty}^{t} E_z^{in}(t', h) \cdot K(t - t', z) dt' \]

\[ K(z, t) = \frac{c}{Z_c} \left[ h(t) - \frac{1}{2} \left[ h(t - (L/2 - z_1)/c) + h(t - (L/2 + z_1)/c) \right] \right] + \\
\quad + 2 \Re \left\{ \sum_{n=1}^{\infty} \frac{1}{j k_n' - k_n''} \cdot \frac{1}{\mu_0 4\pi} \cdot \frac{C(k_n' + jk_n'')}{G(k_n' + jk_n'')} \cdot \frac{L}{1 - e^{-2j(k_n' + jk_n'')h}} \cdot \frac{1}{j(k_n' + jk_n'') \ln(2h/a)} \cdot \left[ \exp\left[ (j k_n' - k_n'') \cdot c \cdot (t - (L/2 - z_1)/c) \right] \cdot h(t - (L/2 - z_1)/c) + \right. \right. \\
\quad \left. \left. \exp\left[ (j k_n' - k_n'') \cdot c \cdot (t - (L/2 + z_1)/c) \right] \cdot h(t - (L/2 + z_1)/c) \right] \right\} \]

Current induced in the center of the open-circuit line (L=10 m, h=0.5 m, a=1 mm, z=5 m) by an unit Gaussian pulse (\(\tau=0.3\) ns). Calculated by:
1. Inverse Fourier transform of NEC result (5000 frequency points in the interval 0\(\leq\omega<10^9\) s\(^{-1}\));
2. Inverse Fourier transform of the asymptotic approach result;
3. Use of SEM response function.
Theory II: SEM and method of modal parameters for thin line of arbitrary geometrical form

*Mixed Potential Integral Equation (MPIE)*

Geometrical characteristics: the wire is represented as a curve with natural parameter $\ell$:

$$\bar{r}(l) = (x(l), y(l), z(l)) \in \{l\}$$

$0 \leq l \leq L/2$, $L/2$ — is the total length of the wire

The unit tangential vector:

$$\bar{e}_t(l) = \frac{\partial \bar{r}(l)}{\partial l}$$

The thin – wire approximation. $a << 2h, \lambda$

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d \varphi(l)}{d l} + j \omega \frac{\mu_0}{4\pi} \int_0^L \mathcal{G}_I^L(l,l') I(l') dl' = E_i^e(l) \\
\int_0^L \tilde{g}_C^I(l,l') \frac{d I(l')}{d l'} dl' + j \omega 4\pi \varepsilon_0 \varphi(l) = 0
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\mathcal{G}_I^l(l,l') &= \frac{e^{-jk \sqrt{(\bar{r}(l)-\bar{r}(l'))^2+a^2}}}{\sqrt{(\bar{r}(l)-\bar{r}(l'))^2 + a^2}} - \frac{e^{-jk \sqrt{(\bar{r}(l)-\bar{r}(l'))^2+a^2}}}{\sqrt{(\bar{r}(l)-\bar{r}(l'))^2 + a^2}} \\
\mathcal{G}_C^l(l,l') &= \frac{e^{-jk \sqrt{(\bar{r}(l)-\bar{r}(l'))^2+a^2}}}{\sqrt{(\bar{r}(l)-\bar{r}(l'))^2 + a^2}} - \frac{e^{-jk \sqrt{(\bar{r}(l)-\bar{r}(l'))^2+a^2}}}{\sqrt{(\bar{r}(l)-\bar{r}(l'))^2 + a^2}}
\end{align*}
\]
Method of modal parameters for the line with arbitrary geometry

\[ \frac{d \varphi(l)}{dl} + j \omega \frac{\mu_0}{4\pi} \int_{\{L\}} \vec{e}_l(l) \cdot \vec{e}_l(l') g(l,l') I(l') dl' = E^e_l(l) \]

\[ \vec{e}_l(l) = \frac{\partial \vec{r}(l)}{\partial l} \]

\[ \int_{\{L\}} g(l,l') \frac{d I(l')}{dl'} dl' + j \omega 4\pi \varepsilon_0 \varphi(l) = 0 \]

Symmetrized Mixed Potential Integral Equation (MPIE)

\[ g(l,l') := \frac{e^{-jk \sqrt{(l-l')^2 + r_0^2}}}{\sqrt{(l-l')^2 + r_0^2}} \]

All functions of natural parameter \( l \) are periodical, i.e., \( \vec{r}(l + L) = \vec{r}(l) \) and they can be expanded in a complete orthogonal set of functions \( e^{-jk_m l} \) with \( k_m := \frac{2\pi m}{L}, m = \ldots -1,0,1\ldots \)

\[ L \text{ is a total length of the loop} \]

\[ \varphi(l) = \sum_{m=-\infty}^{\infty} \varphi_m e^{-jk_m l} = [e^{-jk_m l}]^T \cdot \Phi \]

\[ I(l) = \sum_{m=-\infty}^{\infty} I_m e^{-jk_m l} = [e^{-jk_m l}]^T \cdot I \]

\[ E^e_l(l) = \sum_{m=-\infty}^{\infty} E_{lm} e^{-jk_m l} = [e^{-jk_m l}]^T \cdot E^e_l \]

\[ [e^{-jk_m l}] := [\ldots e^{-jk_{-1} l}, 1, e^{jk_1 l}, \ldots]^T \]
Method of modal parameters (cont.)

\[
\begin{cases}
-jk \cdot \Phi + j\omega L \cdot I = E_l^e \\
-jk \cdot I + jC \cdot \Phi = 0
\end{cases}
\]

MPIE equations in modal representation

\[k = \text{diag}(k_m).
\]

\[
G_L = \begin{bmatrix} G_{m_1,m_2} \end{bmatrix} \quad G_m^{m_1,m_2} = \int_L \int_L \bar{\epsilon}_l(l_1) \cdot \bar{\epsilon}(l_2) g(l_1, l_2) \exp(-jk_{m_2} l_2 + jk_{m_1} l_1)
\]

\[
G_C = \begin{bmatrix} G_{m_1,m_2} \end{bmatrix} \quad G_m^{m_1,m_2} = \int_L \int_L g(l_1, l_2) \exp(-jk_{m_2} l_2 + jk_{m_1} l_1)
\]

Solution of MPIE in modal representation

\[
\begin{cases}
I = Z^{-1} \cdot E_l^e \\
\Phi = \frac{1}{j\omega} C^{-1} \cdot jk \cdot Z^{-1} \cdot E_l^e
\end{cases}
\]

\[Z = \frac{k \cdot C^{-1} \cdot k}{j\omega} + j\omega L = \frac{\eta_0}{4\pi jk} \left[ k \cdot G_C \cdot k - k^2 G_L \right]
\]

\[Z\] is a infinite modal impedance matrix, which defines

a) the scattering field on the boundary of the wire:

\[E^sc_l = -Z \cdot I\]

b) Radiation of the system:

\[W = \frac{L}{4} \text{Re} \left\{ I^+ \cdot Z \cdot I \right\}\]
Equation for SEM poles

\[ I = Z^{-1} \cdot E_l^e = \left[ \frac{k \cdot C^{-1} \cdot k}{j \omega} + j \omega L \right]^{-1} \cdot E_l^e = 4 \pi \varepsilon_0 j \omega \left[ k \cdot G_C \cdot k - k^2 G_L \right]^{-1} \cdot E_l^e \]

The SEM poles are independent from the way of excitation of the system: distributed excitation, lumped excitation, etc.

By this way, the SEM poles are defined by the equation:

\[ \text{det}(Z) = 0 \]

\[ \text{det}[k \cdot G_C \cdot k - k^2 G_L] = 0 \]

\[ \text{det}[k^2 1 - G_L^{-1}(k) \cdot k \cdot G_C(k) \cdot k] = 0 \]
Equation for SEM poles. Symmetrical case

For the partial case of the semi-circular wire (circular total loop, structure with high symmetry - Curvature: \( K=1/R=\text{const} \); Torsion: \( T=0 \). ) the inductance, capacitance and impedance matrices are diagonal and we obtain the solutions for the current and SEM poles.

\[
L_{m_1,m_2}(k) = S_0 R \left( g_{m_1+1} + g_{m_2-1} \right) \cdot \delta_{m_1,m_2} ; \quad C_{m_1,m_2}(k) = \frac{2 \varepsilon_0}{Rg_{m_1}} \cdot \delta_{m_1,m_2} ;
\]

\[
Z_{m_1,m_2}(j\omega) = j\omega L_{m_1,m_2} + \frac{k_m^2}{j\omega C_{m_1,m_2}} \sim \delta_{m_1,m_2} \quad k_m = \frac{m}{R}, \quad m = \ldots, -1, 0, 1, \ldots
\]

\[
g_m = \int_0^{2\pi R} e^{jm\phi - jk \sqrt{4R^2 \sin^2 (l/2R) + a^2}} \sqrt{4R^2 \sin^2 (l/2R) + r_0^2} \, dl
\]

\[
k^2 - \frac{2k_m^2 g_m(k)}{g_{m+1}(k) + g_{m-1}(k)} = 0 ; \quad k^2 - \frac{2k_m^2 g_m(k)}{g_{m+1}(k) + g_{m-1}(k)} = 0 ;
\]

Exact equation for the SEM poles.
Equation for SEM poles, iterative solution

For the case of an infinitely thin wire \( r_0 \to 0 \) the matrices \( G_L \) and \( G_C \) become diagonal and inverse to each other

\[ G^{-1}_L(k) \cdot k \cdot G_C(k) \cdot k \to k^2 \]

which leads to

\[ k^2 = k_m^2 \Rightarrow k_m^{(0)} = \pm 2\pi m/L, \quad m = \ldots -2, -1, 0, 1, 2, \ldots \]

This value gives the zero-order approximation for the perturbation solution:

The first order solution for the roots of the first layer is given:

\[
\det[\text{diag}\left((k_m^{(1)})^2 - G^{-1}_L(k_m^{(0)}) \cdot k \cdot G_C(k_m^{(0)}) \cdot k\right)] = 0
\]
SEM and method of modal parameters for arbitrary line (cont.)

Numerical realization.

1. Calculation of the matrices $GL$ and $GC$ of the order $M_{\text{max}}$ (the poles are taken up to $M<=M_{\text{max}}-2$ and the definition of the determinant $\det(Z(x))$ as a function of the normalized complex wave number $x:=k^*L/2\pi$.

2. As zero-order iteration for the SEM poles of the first layer we use: $x_m^{(0)} = m$

3. The iteration procedure uses the Newton algorithm. Because the calculation of double integrals for all elements $(2M_{\text{max}}+1)*(2M_{\text{max}}+1)$ requires a long time, the iteration procedure will be carried out in two steps for the function $F(\tilde{x},x) := \det[\tilde{k}^2 \cdot GL(k) - k \cdot GC(k) \cdot k]$ where $\tilde{x} := \tilde{k} \cdot L_{\text{max}} / 2\pi$

4. The calculation is carried out in two circles. In the internal circle (15 iterations) we use the Newton algorithm only for the value $\tilde{x}$ under fixed value $x$.

$$\tilde{x}_m^{(j)} := \tilde{x}_m^{(j-1)} - \frac{\partial}{\partial \tilde{x}} F(\tilde{x}_m^{(j-1)}, x_m^{(i)}) \bigg|_{\tilde{x} = \tilde{x}_m^{(j-1)}}$$

5. Next, for external circle we repeat this procedure for $x_m^{(i+1)} = \tilde{x}_m^{(I_{\text{max}})}$, up to $I_{\text{max}} = 3$.

Both initial values are $x_m^{(0)} = \tilde{x}_m^{(0)} = m$
SEM and method of modal parameters. Numerical examples.

1. Checking of the method. Semi-circular wire above PEC.

Short - circuited semi-circular wire:
R=4 m, r₀=1 cm

One can explicitly observe a diagonal character of the modal matrixes for symmetrical line.
SEM poles for circular wire. Comparison of the modal parameter method poles with the exact value. Parameters of the circular loop: R=4 m, r₀=1 cm.

<table>
<thead>
<tr>
<th>Number of the pole m</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tbody>
<tr>
<td>&quot;Exact&quot; solution</td>
<td>Re(xₘ)</td>
<td>1.036</td>
<td>2.050</td>
<td>3.0593</td>
<td>4.0670</td>
</tr>
<tr>
<td>with gₘ</td>
<td>Im(xₘ)</td>
<td>7.00⋅10⁻²</td>
<td>1.0077⋅10⁻¹</td>
<td>1.2415⋅10⁻¹</td>
<td>1.4388⋅10⁻¹</td>
</tr>
<tr>
<td>Method of modal</td>
<td>Re(xₘ)</td>
<td>1.0342</td>
<td>2.0446</td>
<td>3.0514</td>
<td>4.0565</td>
</tr>
<tr>
<td>parameters</td>
<td>Im(xₘ)</td>
<td>6.988⋅10⁻²</td>
<td>1.0029⋅10⁻¹</td>
<td>1.2333⋅10⁻¹</td>
<td>1.3912⋅10⁻¹</td>
</tr>
</tbody>
</table>
SEM and method of modal parameters for arbitrary line
Numerical examples.

2. Semi-elliptical line.

Properties of the matrixes of modal parameters:

\[
\begin{align*}
(G_C)_{-m_1,-m_2} &= (G_C)_{m_1,m_2} \\
(G_C)_{m_2,m_1} &= (G_C)_{m_1,m_2} \\
(G_L)_{-m_1,-m_2} &= (G_L)_{m_1,m_2} \\
(G_L)_{m_2,m_1} &= (G_L)_{m_1,m_2}
\end{align*}
\]

Ellipse: \(a=4\) m, \(b=1\) m, \(r_0=1\) cm, \(N_{\text{max}}=7\)
Ellipse

a = 4 m, b = 1 m, r₀ = 1 cm

N_max = 7

SEM poles for ellipse
a_ellips = 4 m, b_ellips = 1 m
r₀ = 1 cm

Re(G_C)

Im(G_C)
SEM and method of modal parameters for arbitrary line
Numerical examples.

3. Horizontal wire with vertical risers ("Vance configuration").

\[ L = 10 \text{ m}, \ h = 0.5 \text{ m}, \ r_0 = 1 \text{ cm} \]
3. Horizontal wire with vertical risers (“Vance configuration”), cont.

Re($G_c$), Vance configuration, $n_{eff}=2.3$

Im($G_c$), Vance configuration, $n_{eff}=2.3$

Function $1/|F(x,x)|$ for the straight horizontal wire with short-circuited risers

NEC

Asymptotic approach

Method of modal parameters

Re($k_n$)($L+2^{*}h$)/$\pi$

Im($k_n$)($L+2^{*}h$)/$\pi$
Description of the results.

• The agreement with „exact“ numerical results, as well as with the approximate analytical method for the Vance configuration is much better than in the case of elliptic wire. This may be caused by the uniform length division in this case, unlike the case of ellipsoid.

Differential (a) and common (b) modes for the Vance structure.

• Another important observation is related to the excitation of two types of modes - common mode and differential mode. The differential mode \((n=0,2,4,...,\) see Fig. a) appears for any transmission line structure, which has horizontal components. The common mode \((n=1,3,5,...,\) see Fig. b) appears for the transmission line structure, which has vertical elements and exiting, for example, by distributed source, which has vertical components. Note, that both types of modes, of course, appear in two previous examples: semi-circular wire and semi-elliptical wire above perfect conducting ground with non-symmetrical excitation.

In the case of wire without vertical component (open-circuit wire above a ground), or in the case of excitation of symmetrical configurations (all three considered configurations) by the symmetrical way (i.e., plane wave with normal incidence or grazing incidence) we can observe only one type of modes (differential modes or common modes).
Measurements and numerical simulation

The measurement of the field coupling was done in a GTEM cell (septum height of 1.75 m, maximum test volume of ~1 m³, usable frequency range from DC to 18 GHz). With the help of a vector network analyzer (VNA, Rohde&Schwarz ZVM, 10 MHz to 20 GHz) the scattering parameters between the base point of the wire (port 1 of the VNA) and the feeding port of the GTEM cell (port 2 of the VNA) were measured.

The frequency response of the line was simulated using MoM NEC (Numerical Electromagnetic Code) software.

In both cases the SEM poles can be extracted from the data using a Padé approximation as described by T.B.A. Senior and J.M. Pond.
Measurements and numerical simulation
Horizontal open-short circuit wire

\[
\hat{E} \quad \hat{k}
\]

\[h = 1.5 \text{ cm}, \quad L = 30 \text{ cm}, \quad d = 0.8 \text{ mm}\]

Measurements in TEM cell

\[|I(0, \omega)|, \text{ A}\]
\[k^*(h+L)/\Pi\]

NEC calculation
Grazing incidence
\[E_0 = 1 \text{ V/m}\]

NEC calculated current for grazing incidence of the exciting wave

\[|\Phi(0, \omega)|, \text{ 1/Ohm}\]
\[l^*(L+h)/\Pi\]

Measurements in TEM cell

\[|I(0, \omega)|, \text{ A}\]
\[k^*(L+h)/\Pi\]

Re\((k^*n(L+h)/\Pi)\)

Im\((k^*n(L+h)/\Pi)\)
Conclusion

• We presented different analytical techniques to define SEM poles of thin-wires structures.

• The SEM poles of the first layer for a horizontal long transmission line above perfect conducting ground were investigated using a previously developed asymptotic approach method.

• The SEM poles of the wire of arbitrary form were investigated using previously developed method of modal parameters.

• SEM expansion allows to obtain results in time domain and curried out identification of wiring systems.

• Obtained results for some important configurations were compared with measurements and simulation results and a good agreement was achieved.

• Both used methods allow to consider loaded wires.

• In the future we would like to simplify the modal parameters method, using an approach developed earlier by J.M. Myer, S.S. Sandler and T.T. Wu for a straight wire.
References